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Non-Trivial Coupling of Internal and Space-Time Symmetries

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Abstract

Previous results which exhibit the impossibility of combining internal and space-time symmetries are reanalysed. Starting with McGlinn's (1964) work, where this trivial coupling appears as a direct product of these symmetries, it is shown that, by suitable generalisations such as the use of the framework of group extensions, and by the introduction of a new quantal observable, i.e. a 'mass-breaking operator', one can prove the existence of a non-trivial coupling scheme which admits mass splitting for the members of some super-multiplet. This leads to a new classification scheme for elementary particles. Contrary to the conventional classification models, where the choice of the underlying symmetry group does not emerge directly from the comparison of theoretical predictions with the experimental data, our scheme admits the possibility of determining the relevant symmetry from the mass spectrum.

1. *Introduction*

The aim of this article is to reappraise McGlinn's Theorem (McGlinn, 1964) as well as some subsequent works, which afforded negative results with regard to the coupling of space-time and the so-called internal symmetries. We shall analyse in particular what kind of ingredients, viz. modifications of the aforementioned works, which refer to semi-simple or simple symmetries like $SU(n)$, are required in order to yield a non-trivial coupling scheme. That is, we shall analyse in Section 3 which class of 'higher' symmetries provides a grouping of a given set of particles and admits at the same time a solution of the problem of combining with relativistic invariance in a non-trivial way. Otherwise stated: We tackle the problem of whether there exist such non-trivial couplings of space-time symmetry $\{(a, \Lambda)\}$ and 'internal symmetries' at all and how these symmetries are then related to each other in some suitable framework.

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2. *Previous Results on Trivial Couplings of Internal and Space-Time Symmetries*

Let us agree to call internal symmetries those transformations which act on the 'internal parameters', such as isospin, hypercharge, baryonic number, etc., of a particle. These symmetries must be considered to be violated by small or large amounts in order to yield predictions which agree with experimental data. That is, the particles of some multiplet differ in mass, and this mass splitting is quite considerable in the case of symmetries as for instance the unitary symmetry $SU(3)$.

This feature of symmetries which are not exact already appears in non-relativistic quantum mechanics. This is, for example, the case when an external electric field (Stark effect) violates the rotation symmetry of some Hamiltonian, H_0, with a spherically symmetric potential, that is

$$H = H_0 - eEz \tag{2.1}$$

and where the only constant of motion remains the generator I_z of $SO(3)$. This symmetry-breaking mechanism was for a long time also the accepted picture in elementary particle physics, i.e. the mass splitting was explained by the assumption that the symmetry is broken by some interaction Hamiltonian. The breaking of the Gell–Mann–Neeman $SU(3)$-symmetry accounts for this. A mass splitting such that $SU(3)$ is broken but yet isospin and hypercharge are still conserved, entails the theory to be invariant under transformations corresponding to the smaller group

$$SU(2)_I \times U(1)_Y \subset SU(3) \tag{2.2}$$

where $SU(2)_I$ denotes the isospin group, and $U(1)_Y$ the hypercharge gauge group. If one assumes again that a component of H violates the fundamental symmetry, one has to write, in analogy with (2.1):

$$H = H_0 + H_1 \tag{2.3}$$

where H_0 is invariant under $SU(3)$. According to (2.2), H_1 must be charge independent and hypercharge conserving, i.e.

$$[H_1, I_k] = [H_1, Y] = 0, \qquad k = 1, 2, 3 \tag{2.4}$$

This picture of broken symmetries is, however, very unsatisfactory. Indeed, the 'switching on' of the symmetry-breaking part of the interaction entails a change in the mass values and consequently yields inequivalent representations of the inhomogeneous Lorentz group $\{(a, \Lambda)\}$, which are labelled by the spin and mass values. This transition certainly constitutes a non-trivial mathematical procedure.

The approach advocated by McGlinn circumvents these problems but leads only to a trivial coupling scheme. McGlinn tries to explain the mass splitting within multiplets by purely group theoretical methods. The physical starting point is that within a multiplet of an internal interaction symmetry group G_1 all particles possess the same spin and parity. This means that these quantum numbers must be invariant under the internal symmetry transformations:

$$[M_{\mu\nu}, X_k] = 0, \qquad k = 1 \ldots \dim G \tag{2.5}$$

where $M_{\mu\nu} = -M_{\nu\mu}$ denote the covariant components of the angular momentum tensor and X_k the infinitesimal generators of the Lie algebra $\mathscr{G}_1$ of G_1. But the internal quantum numbers may not be translation invariant, since the particles in a multiplet have different masses. Therefore

$$[P^\mu, X_k] \neq 0 \tag{2.6}$$

McGlinn has proved that if (2.5) holds $\forall X_k \in \mathscr{G}_1$, the semi-simple Lie algebra of G_1, then all these infinitesimal operators commute with every infinitesimal operator of the translations $\{(a, I)\}$. If, furthermore, $\mathscr{G}(P)$ (the Lie algebra of the inhomogeneous Lorentz-transformations) and $\mathscr{G}_1$, the internal algebra, are by assumption sub-algebras of some overall symmetry Lie algebra $\mathscr{G}(G)$, then this means:

$$\mathscr{G}(G) = \mathscr{G}(P) \oplus \mathscr{G}_1 \qquad \text{(direct sum)}, \tag{2.7}$$

or equivalently

$$G = P \times G_1 \qquad \text{(direct product)} \tag{2.7$'$}$$

which yields a degenerate mass spectrum and thus invalidates (2.6).

In view of this negative result of McGlinn one is led to consider symmetry schemes which generalize the direct product structure (2.7′). An interesting attempt has been put forward by Michel (1965a). Instead of the requirement (2.5) or equivalently the statement, that for every $g \in G_1$ and every $p \in P$, $gp = pg$, Michel's Lemma assumes the minimum hypothesis to hold, that there is at least one Lorentz transformation that commutes with G.

Lemma 1 [Michel (1965a)]

Let G be any symmetry group and let the internal symmetry G_1 and P be analytic subgroups, such that

$$P \cap G_1 = \{1\} \tag{2.8}$$

$$G = G_1 \cdot P = \{g = g_1 p : g_1 \in G_1, p \in P\} \tag{2.9}$$

if there exists at least one Lorentz transformation

$$\Lambda_0 \in \{(0, \Lambda)\} \tag{2.10}$$

that commutes with G_1, then G is a semi-direct product of P and G_1 denoted by

$$G = P \circledS G_1. \tag{2.11}$$

Remark 1: Otherwise stated: If *one* infinitesimal generator of a given semi-simple Lie group commutes with all generators of the homogeneous Lorentz group, then the combined group is necessarily the semi-direct product.

Remark 2: If one assumes furthermore that G_1 commutes with $\{(0,\Lambda)\}$ and if G_1 is supposed to be semi-simple, this entails McGlinn's Theorem for groups, i.e. formula (2.7′).

We give now a short proof of Michel's Lemma, which is particularly simple [see also von Westenholz (1970a) and Hermann (1966)]:

Proof of Michel's Lemma

It is sufficient to show that $P \subset G$ be an invariant subgroup in order to obtain (2.11). Consider the homogeneous space

$$G/P = \{g_1 P: g_1 \in G_1\} \tag{2.12}$$

and define

$$I = \{i \in G: i(g_1 P) = g_1 P \ \forall \ g_1\} \tag{2.13}$$

that is, the set (2.13) consists of those elements of G that act trivially on the coset space (2.12). The proof then consists in showing that

(a) I is an invariant subgroup of G and that,
(b) $P \subseteq I$.

Proof of step (a):

$$gig^{-1}(g_1 P) = i'(g_1 P) = g_1 P \qquad i' \in I,$$

since

$$gi(g^{-1} g_1 P) = g(g^{-1} g_1 P)$$

Proof of step (b):

$$\Lambda_0(g_1 P) = (\Lambda_0 g_1)P = (g_1 \Lambda_0)P = g_1(\Lambda_0 P) = g_1 P \ \forall \ g_1$$

Therefore $\Lambda_0 \in I$. Since P has no non-trivial invariant subgroup except $\{(a,I)\}$ one finds $P \cap I = P$.

However, semi-direct couplings which constitute a first generalization of direct products do not provide non-trivial couplings and therefore do not account for mass splitting. This point is discussed in Hegerfeldt & Hennig (1967).

Another attempt to explain mass splitting within a group theoretical framework is due to O'Raifeartaigh (1965), Jost (1966) and Segal (1967). Their approach consists essentially in showing that if the inhomogeneous Lorentz group P is imbedded in a higher symmetry group, then the mass operator $M^2 = P_\mu \cdot P^\mu$ has one and only one discrete eigenvalue. Therefore, no mass splitting with discrete masses is possible with *finite*-dimensional symmetry groups within the framework of embeddings.

In view of these negative results with regard to mass splitting by finite-dimensional Lie groups and algebras, attempts have been made in order to explain mass splitting by infinite-dimensional algebras or groups. The attempt made by the author (1970b) uses suitably parametrized non-denumerable infinite-parameter Lie groups, the so-called 'nuclear' Lie groups, which have been studied by Gelfand (1964). Within the framework of such symmetries and the conventional assumptions about the mass spectrum one obtains again only trivial couplings. Another approach, which is advocated by Formanek (1966), uses real Lie algebras of denumerable infinite dimension. With sufficiently restrictive assumptions one can force mass splitting with such symmetries. However, it has been shown by Flato & Sternheimer (1966), that infinite-dimensional Lie groups of the type as proposed by Formanek yield the non-unique result that any mass formula can be obtained. Indeed, let $U(m_i, s)$ ($i = 1 \ldots 8$, m_i denote the eight masses of the baryon octet with spin $s = \frac{1}{2}$) be continuous inequivalent unitary representations of $\{(a, \Lambda)\}$. Then

$$U_1(a, \Lambda) = \bigoplus_{i=1}^{8} U(m_i, \tfrac{1}{2}) : \mathscr{H} \to \mathscr{H} \tag{2.14}$$

operates on the separable Hilbert space $\mathscr{H}$.

Let $\{\varphi_{ik}/k = 1, 2, \ldots\}$ be a countable orthonormal basis in the spaces $\mathscr{H}_i$ such that:

$$\begin{matrix} \{\varphi_{11}\,\varphi_{12}\ldots\varphi_{1\aleph_0}\} \in \mathscr{H}_1 \\ \vdots \quad \vdots \qquad \vdots \\ \{\varphi_{81}\,\varphi_{82}\ldots\varphi_{8\aleph_0}\} \in \mathscr{H}_8 \end{matrix} \tag{2.15}$$

i.e. $\dim\mathscr{H}_1 = \ldots \dim\mathscr{H}_8 = \aleph_0$, that is $\mathscr{H}_1 \cong \mathscr{H}_\kappa$, $k, i = 1 \ldots 8$. For any fixed k the vectors $\varphi_{1k} \ldots \varphi_{8k}$ generate an 8-dimensional Hilbert space $\mathscr{H}_k'$ on which one can define a unitary representation $Ad_k(SU(3))$ of $SU(3)$. Therefore:

$$\mathscr{H} = \bigoplus_{k=1}^{\aleph_0} \mathscr{H}_k{}^1, \qquad \dim \mathscr{H}_k{}^1 = 8 \tag{2.16}$$

carries the representation

$$U_2(SU(3)) = \bigoplus_{k=1}^{\aleph_0} Ad_k(SU(3)) \tag{2.17}$$

of $SU(3)$. Consider now

$$U_2 : SU(3) \to \{U/U : \mathscr{H} \to \mathscr{H}\} := GL(\mathscr{H}) \tag{2.18}$$

to be an injective mapping of $SU(3)$ into the group of all unitary operators acting on $\mathscr{H}$. According to Maissen (1962) this group of continuous automorphisms of a Banach space has the structure of a Lie group. Since evidently $\{U_1(a, \Lambda)\} \subset GL(\mathscr{H})$, one has obtained a $SU(3)$ classification and a mass splitting by mixing representations of $SU(3)$ and $\{(a, \Lambda)\}$, both contained in the infinite-dimensional Banach-Lie group $GL(\mathscr{H})$. As this construction goes through for any group and any number of multiplets, such pre-assigned masses could originate from any mixing and thus from any mass formula.

In summary, in the use of infinite-dimensional symmetry groups one introduces the following severe difficulties:

(1) One introduces a multitude of new quantum numbers which have not yet been observed.
(2) One is faced with the aforementioned problem of the non-uniqueness of the resulting mass formulae.
(3) The only invariant scattering operator S is the trivial one, i.e. $S = I$. This can be shown within the context of the aforementioned 'nuclear' infinite-parameter groups [von Westenholz (1969)], but this result should hold for any infinite-parameter group. The reason for this is roughly that with an infinite parameter group, one introduces an infinite number of conserved commuting quantum numbers. Conservation of all these quantities restricts the S-matrix to such an extent that no scattering is possible.

3. *Description of a Non-Trivial Coupling Scheme Between Internal and Space-Time Symmetries*

As a starting point, we remark that the direct product (2.7′) $G_1 \times P_+^\uparrow$ ($P_+^\uparrow$ stands for the orthochronous inhomogeneous Lorentz group with determinant +1) constitutes a special case of a group extension. Group extensions have been studied extensively by Eilenberg & MacLane (1947) and Eilenberg (1949) in the general case and by Michel (1962, 1965b) for the special case of the inhomogeneous Lorentz group.

The problem of group extensions can be posed as follows: Given two groups, K and Q, find all groups E such that

$$\text{(a)}\quad K \lhd E \qquad (K \text{ is an invariant subgroup of } E) \tag{3.1}$$

and

$$\text{(b)}\quad Q = E/K$$

E is then called an extension of Q by K. The group extensions of the type (3.1) may be written diagrammatically as a short exact sequence

$$\{e\} \xrightarrow{f_0} K \xrightarrow{f_1 = i} E \xrightarrow{f_2 = \phi} Q \xrightarrow{f_3} \{e\} \tag{3.2}$$

where $\{e\} := 1$ denotes the trivial group which consists of the neutral element alone. The f_i are homomorphisms. (A sequence $\ldots \xrightarrow{f_i} G_i \xrightarrow{f_{i+1}} \ldots$ is called exact if $\mathrm{Ker}(f_{i+1}) = \mathrm{Im}(f_i)\ \forall_i$.)

Remark 3: Such a short exact sequence expresses the relationship $Q = E/K$ since one has, by considering the general diagrams,

$$1 \to A \to B \to C \to 1 \tag{3.3}$$

$$1 \longrightarrow A \xrightarrow{f_1} B \tag{3.4}$$

means that $\mathrm{Ker}(f_1) = 1$, i.e. f_1 is an injection map and thus $f_1(A) \cong A$ and

$$B \xrightarrow{f_2} C \xrightarrow{f_3} 1 \tag{3.5}$$

expresses that f_2 is surjective, i.e. $f_2(B) = C$ (since $\mathrm{Ker}(f_3) = C$). Furthermore we have $B/\mathrm{Ker}(f_2) \cong f_2(B)$ and therefore $B/A \cong C$ since $\mathrm{Im}(f_1) = \mathrm{Ker}(f_2)$.

The direct product (2.7′) as well as the semi-direct product (2.11) may, in terms of group extensions, be written by means of the following special short exact sequence (3.6) which is said to split:

$$1 \longrightarrow G_1 \xrightarrow{i} G \underset{u}{\overset{\phi}{\rightleftarrows}} P_+{}^\uparrow \longrightarrow 1 \tag{3.6}$$

That is: There exists an injective homomorphism u: $P_+{}^\uparrow \to G$ such that $\phi_0 u = 1_{P_+\uparrow}$ represents the identity automorphism of $P_+{}^\uparrow$.

Upon identification of G_1 with $i(G_1)$ and $u(P_+{}^\uparrow)$ with $P_+{}^\uparrow$ one obtains: $G_1 \lhd G$, $P_+{}^\uparrow$ is a subgroup of G such that $G_1 . P_+{}^\uparrow = G$ and $G_1 \cap P_+{}^\uparrow = \{e\}$. This case corresponds to the semi-direct product (2.11). If, in addition $P_+{}^\uparrow \lhd G$, then this yields (2.7′).

However, in general, u will not be a homomorphism and the deviation from the law of homomorphism is a map

$$f\colon P_+{}^\uparrow \times P_+{}^\uparrow \to G_1 \tag{3.7}$$

which has the property

$$u(L_r)\, u(L_s) = f(L_s, L_s) . u(L_r . L_s), \qquad L = (a, \Lambda) \in P_+{}^\uparrow \tag{3.8}$$

In order to display the existence of non-trivial couplings between space-time and internal symmetries one may generalise the relationship (3.6) within the framework of cohomology theory (Eilenberg & MacLane, 1947; Eilenberg, 1949) as follows: Let

$$C^0 \xrightarrow{\delta_0} C^1 \xrightarrow{\delta_1} \dots \xrightarrow{\delta_{n-2}} C^{n-1} \xrightarrow{\delta_{u-1}} C^n \xrightarrow{\delta_u} C^{n+1} \tag{3.9}$$

be a sequence of abelian groups $C^k = C^k(P_+{}^\uparrow, G_1)$ where the homomorphisms are such that $\delta_{k+1}\delta_k = 0$. Then we define n-dimensional cocycles $f \in Z^n$, coboundaries $b \in B^n$ and cohomology classes $\{f\} \in H^n$ by the following expressions:

$$Z^n = \{f \in C^n\colon \delta f = e \in C^{n+1}\}\colon \qquad \text{Kernel of } \delta_n\colon C^n \to C^{n+1} \tag{3.10}$$

$$B^n = \{b \in C^n\colon b = \delta f, f \in C^{n-1}\}\colon \quad \text{Image of } \delta_{n-1}\colon C^{n-1} \to C^n \tag{3.11}$$

and

$$H^n = Z^n / B^n\colon \quad n\text{th cohomology group} \tag{3.12}$$

In formula (3.12), two cases must be distinguished:

$H^n = 0$: The sequence (3.9) is exact, i.e. $\mathrm{Im}\,\delta_n = \mathrm{Ker}\,\delta_{n+1}$ (3.13a)

$H^n \neq 0$: The nth cohomology group measures the lack of exactness of the sequence (3.9) (3.13b)

In his paper Eilenberg (1947) gives a result connecting group extensions and cohomology and which is, expressed by means of (3.2) and (3.13), that the group $\mathrm{Ext}_g(P_+{}^\uparrow, G_1)$ of extensions of $P_+{}^\uparrow$ by G_1 is isomorphic to the second cohomology group $H^2(P_+{}^\uparrow, G_1)$. (g denotes the homomorphism $P_+{}^\uparrow \to \mathrm{Aut}\, G_1/\mathrm{Int}\, G_1$, refer to the diagram (3.31) in our subsequent discussion). G_1 is supposed to be an abelian group, but it turns out that our scheme is readily generalised for non-abelian groups as shown in von Westenholz (1971a).

Our next task is to construct a non-trivial coupling scheme in which the masses of a supermultiplet are related to the extensions $\mathrm{Ext}(P_+{}^\uparrow, G_1)$ in such a way, that particularly the mass-degeneracy is associated with the direct product coupling (2.7′), i.e. (3.6). This can be achieved as follows: Let $\mathscr{H}^{\otimes 1}$ be the single particle Hilbert space. In the case where particles within a multiplet have different masses $m_1, m_2 \ldots m_n$, $\mathscr{H}^{\otimes 1}$ has necessarily to be written as the direct sum of the subspaces $\mathscr{H}_j^{\otimes 1}, j = 1 \ldots n$,

$$\mathscr{H}^{\otimes 1} = \bigoplus_{j=1}^{n} \mathscr{H}_j^{\otimes 1} \tag{3.14}$$

where

$$\mathscr{H}_j^{\otimes 1} := \mathscr{H}(m_i, s) \qquad (s \text{ denotes the spin of the } n \text{ particles}) \tag{3.15}$$

Then, a corollary of Schwartz's Kernel Theorem (Schwartz, 1960) states that there exists a bijective correspondence between these Hilbert spaces and a family of positive Kernel-distributions in two variables $(K_{xy}^i)_{1 \leqslant i \leqslant n}$

$$K_{xy}^i \leftrightarrow \mathscr{H}(m_i, s) \qquad (i = 1 \ldots n) \tag{3.16}$$

A closer inspection of the corresponding Fourier-transformed positive measures

$$\mathscr{F} K_{xy}^i = \mu_i = c\delta(p_i^2 - m_i^2) \qquad (i = 1 \ldots n) \tag{3.17}$$

reveals that one is led to distinguish the following two cases:

$$m_1 = m_2 = \ldots = m_n \qquad \text{i.e. } \triangle m_{ij} = 0 = m_i - m_j, \tag{3.18a}$$

the measures (3.17) are identical $\forall_i$

$$m_1 \neq m_2 \neq \ldots . \neq m_n \qquad \text{i.e. } \triangle m_{ij} = m_i - m_j \neq 0 \qquad (i > j) \tag{3.18b}$$

$$(3.18\text{a, b}) \Rightarrow (K_i - K_j)\varphi := K_{ij}\varphi = \psi^{ij} \in K_{ij}\mathscr{D} \subset \mathscr{D}'(\mathbf{R}^4, \mathscr{H}_{G_1}) \tag{3.19}$$

($\mathscr{D}'(\mathbf{R}^4, \mathscr{H}_{G_1})$ denotes the space of vector-valued distributions on ($\mathbf{R}^4$).

Thus from (3.18a) and (3.18b) we have the following correspondence

$$K_{ij} \leftrightarrow \triangle m_{ij} \qquad \text{for the } n \text{ independent } \triangle m \tag{3.20}$$

The aforementioned classification of the mass differences, equation (3.18), may now be related to the cohomology classes of $H^2(P_+{}^\uparrow, G_1)$ by means of the wave-distributions (3.19) and the following expression

$$U(f(L_r, L_s))\,\phi_0 = \psi_{(L_r, L_s)}^{ij}(\varphi_0) := \int_{\mathbf{R}^4} \psi^{ij}(L_r L_s x)\,\varphi_0(x)\,dx = \phi \in \mathscr{H}_{G_1} \tag{3.21}$$

where

$f \in Z^2(P_+{}^\uparrow, G_1)$,

$(L_r, L_s) \in P_+{}^\uparrow \times P_r{}^\uparrow$, $x \in \mathbf{R}^4$,

$\phi_0 \in \mathscr{H}_{G_1}$ (representation space of the internal symmetry G_1),

$\varphi_0 \in \mathscr{D} = \{\varphi \in C^\infty |$ supp (φ): compact$\}$, and

$\psi^{ij}_{(L_r, L_s)}(\varphi_0)$: vector-valued distribution $\in \mathscr{H}_{G_1}$.

As G_1 is a group by assumption and since one has, according to (3.7),

$$f(L_r, L_s) \in G_1 \ \forall f \in Z^2(P_+{}^\uparrow, G_1), \qquad (L_r, L_s \in P_+{}^\uparrow \times P_+{}^\uparrow$$

one can define, as Z^2 is a group, a binary operation in G_1 as follows:

$$f_i(L_r, L_s) + f_j(L_r', L_s') = (f_i + f_j)(L_r L_r', L_s L_s') = f_k(L_r'', L_s''), \quad (3.22)$$

where $f_k \in Z^2(P_+, G_1)$ and L_r, $L_s \ldots$ fixed.

Now consider the set

$$G_1' := \{b(L_r, L_s) : b \in B^2(P_+{}^\uparrow, G_1), (L_r, L_s) \in P_+{}^\uparrow \times P_+{}^\uparrow\} \qquad (3.23)$$

endowed with the composition law (3.23). G_1' obviously becomes a subgroup of G_1, since $B^2(P_+{}^\uparrow, G_1) \subset Z^2$ is a group. Then we have the following.

Lemma 2

Let $g \to U(g) = U(f(L_r, L_s))$:; $f \in Z^2(P_+{}^\uparrow, G_1)$, be the continuous unitary representation defined by (3.21). Let $(\mathscr{H}_{ij}(G_1'))$ be a family of pairwise orthogonal cyclic subspaces of $\mathscr{H}(G_1')$, given by

$$\mathscr{H}_{ij}(G_1') = \overline{\{\psi^{ij}_{(L_r, L_s)}(\varphi_0) = U(b(L_r, L_s))\, \psi_0{}^{ij}_{(L_r, L_s)}(\varphi_0) : \psi^{ij} \in K_{ij}\mathscr{D} \subset \mathscr{D}'(\mathbf{R}^4, \mathscr{H}_{G_1})\}} \qquad (3.24)$$

with the cyclic vectors $\psi_0^{ij}(\varphi_0)$. The sub-representations $U_1'(i,j)$ of $U' = U/G_1'$ (the restriction of U to the subgroup G_1') are supposed to be defined by the spaces $\mathscr{H}_{ij}(G_1')$. Then, to any given mass difference $\Delta m \neq 0$ corresponds one and only one cohomology class $\{f\}$.

The proof of this Lemma can be achieved in two steps:

(a) Due to the assumed cyclicity, which obviously entails

$$\mathscr{H}_{G_1'} = \bigoplus_{i,j} \mathscr{H}^{ij}(G_1')$$

one has

$$K_{ij}\mathscr{D} \cap K_{lm}\mathscr{D} = \{0\} \qquad (3.25)$$

which means the disjointedness of the Kernel-distributions associated with different mass differences Δm_{ij} and Δm_{lm}.

(b) Let K_{ij} satisfy (3.20) and let ψ_e^{ij} and ψ_m^{ij} be two elements of $K_{ij}\mathscr{D}$ related to f and $f' \in Z^2(P_+{}^\uparrow, G_1)$ by the definition (3.21). Then one readily shows that

$$f \sim f' \text{ mod } B^2(P_+{}^\uparrow, G_1) \text{ holds} \qquad (3.26)$$

Indeed:

$$\psi^{ij}_{l(L_r,L_s)}(\varphi_0) = U(f(L_r, L_s))\,\phi_0 = U(b_l(L_r, L_s))\,\psi^{ij}_{0(L_r,L_s)}(\varphi_0)$$
$$\psi^{ij}_{m(L_r,L_s)}(\varphi_0) = U(f'(L_r, L_s))\,\phi_0 = U(b_m(L_r, L_s))\,\psi^{ij}_{0(L_r,L_s)}(\varphi_0)$$

But since

$$\psi^{ij}_{0(L_r,L_s)}(\varphi_0) = U(f_0(L_r, L_s))\,\phi_0$$

this yields

$$\begin{matrix} f = b_1 + f_0 \\ f' = b_m + f_0 \end{matrix} \quad \Rightarrow \tag{3.26}$$

However, statement (3.26) does not necessarily exclude the following:

$$\text{and} \qquad \begin{matrix} \psi^{ij} \in K_{ij}\mathscr{D} \\ \psi^{lm} \in K_{lm}\mathscr{D} \end{matrix} \quad \overset{(3.21)}{\Rightarrow} \quad f \sim f' \bmod B^2(P_+{}^{\uparrow}, G_1) \tag{3.26$'$}$$

thus invalidating Lemma 2. The following Lemma 3 ensures that this cannot occur.

Lemma 3

If $f \in Z^2(P_+{}^{\uparrow}, G_1)$ corresponds to $\psi^{ij} \in K_{ij}\mathscr{D}$ and f' corresponds to $\psi^{lm} \in K_{lm}\mathscr{D}$ with the property (3.25) then

$$f \nsim f' \bmod B^2(P_+{}^{\uparrow}, G_1)$$

The proof of this can be found in von Westenholz (1971). The converse of this Lemma is also true (von Westenholz, 1971), since

$$f \nsim f' \bmod B^2(P_+{}^{\uparrow}, G_1) \quad \overset{(3.21)}{\Rightarrow} \quad \begin{matrix} \psi^{ij} \in K_{ij}\mathscr{D} \\ \psi^{lm} \in K_{lm}\mathscr{D} \end{matrix} \tag{3.27}$$

In conclusion, the relationships (3.25), (3.26) and (3.27), in conjunction with Lemma 3, yield Lemma 2. In particular one can show (refer to our subsequent Lemma 4) that vanishing mass differences must be associated with the cohomology class $\{0\} \in H^2(P_+{}^{\uparrow}, G_1)$. According to the quoted bijective correspondence, one has

$$H^2(P_+{}^{\uparrow}, G_1) \leftrightarrow \operatorname{Ext}(P_+{}^{\uparrow}, G_1) \tag{3.28}$$

and by virtue of the relationships (3.20) and Lemma 2 one finds

$$\Delta m_{ij} \neq 0 \xleftrightarrow{\text{bijective}} E_{ij} \in \operatorname{Ext}(P_+{}^{\uparrow}, G_1) \tag{3.29}$$

Thus we obtain the following.

Theorem 1

Non-trivial couplings of space-time and internal symmetries are given by the elements $E_{ij} \in \operatorname{Ext}(P_+{}^{\uparrow}, G_1)$. In particular, the trivial direct product-coupling $G = G_1 \times P_+{}^{\uparrow}$ (or equivalently diagram (3.6)) is related to the mass-differences $\Delta m_{ik} = 0\ \forall\ i, k$.

As mentioned already, this Theorem can be generalised for non-abelian groups (refer to von Westenholz, 1971a).

Discussion of Theorem 1

In connection with this theorem one has to analyse the problem of the existence of the assumed extensions. For this purpose, let us select a homomorphism

$$\theta: P_+{}^{\uparrow} \to \operatorname{Out} G_1 \tag{3.30}$$

such that the following commutative diagram holds:

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[d, dashed] & P_+{}^{\uparrow} = G/G_1 \arrow[d, dashed, "\theta"'] \\ \operatorname{Aut} G_1 \arrow[r] & \operatorname{Out} G_1 = \operatorname{Aut} G_1/\operatorname{Int} G_1 \end{tikzcd} \tag{3.31}$$

where $\operatorname{Aut} G_1$ denotes the group of automorphisms, $\operatorname{Out} G_1$ the outer automorphisms and $\operatorname{Int} G_1$ the inner automorphisms of G_1. Then, in terms of diagram (3.31) one may reformulate the extension problem of $P_+{}^{\uparrow}$ (Michel, 1965c). Given the group G_1, $P_+{}^{\uparrow}$ and the homomorphism (3.30), find all extensions G of $P_+{}^{\uparrow}$, with kernel (G_1, θ), such that G operates on its invariant subgroup G_1 according to diagram (3.31). It may occur that the problem of constructing such extensions has no solution for a given θ. However, in the case of the unitary symmetries $SU(n)$, which are compact-simple Lie groups, there always exists a class of extensions of the inhomogeneous Lorentz group $P_+{}^{\uparrow}$. These are the so-called central extensions, which are characterised by the property $\operatorname{Im}\theta = 1$. Michel (1965) has shown that the central extensions of the Lorentz group are those of the form

$$E_\alpha = G \times \bar{P}_+{}^{\uparrow}/\mathbf{Z}_2(\alpha) \tag{3.32}$$

i.e. the quotient of the direct product $G_1 \times \bar{P}_+{}^{\uparrow}$ ($\bar{P}_+{}^{\uparrow}$ stands for the universal covering group of $P_+{}^{\uparrow}$) by a two-element group $\mathbf{Z}_2(\alpha)$. Although all extensions given by (3.32) are inequivalent, they are isomorphic. Furthermore since, by setting $\alpha = 1$, (3.32) yields the direct product-coupling, the extensions (3.32) are isomorphic to the latter, and therefore from a physical point of view without any interest! Galindo (1967) has been able to show that the existence of essential non-central extensions of $P_+{}^{\uparrow}$ by some internal symmetry group implies that this is necessarily non-semi-simple and non-compact. Therefore our mass splitting model admits essential non-central extensions provided the internal symmetry satisfies the aforementioned restrictions.

In order to obtain mass formulas, associated with this generalised mass-splitting model, one may proceed by introducing a new quantal observable A, which we call 'mass-breaking' operator, and which displays the properties listed below:

$$A \in \mathscr{L}\left(\bigoplus_{n=0}^{\infty} \mathscr{H}^{\otimes n}, \bigoplus_{n=0}^{\infty} \mathscr{H}^{\otimes n}\right) \tag{3.33}$$

where

$$\left.\begin{aligned}
&\bigoplus_{n=0}^{\infty} \mathscr{H}^{\otimes n} = \{\phi = (\phi_0, \phi_1, \ldots \phi_k, \ldots) | \phi_k \in \mathscr{H}^{\otimes k}, \sum_{k=0}^{\infty} \|\phi_k\|^2 > \infty\} \\
&\mathscr{H}^{\otimes 0} := \langle \Omega \rangle = \{c\Omega / c \in \mathbf{C}\}: \text{ the vacuum state} \\
&\mathscr{H}^{\otimes 1} \subset \mathscr{D}'(\mathbf{R}^4, \mathscr{H}_{c_1}): \text{ single particle space} \\
&\mathscr{H}^{\otimes k} = \mathscr{H}^{\otimes 1} \underbrace{\otimes \ldots \otimes}_{k} \mathscr{H}^{\otimes 1}
\end{aligned}\right\} \tag{3.34}$$

In terms of A the mass differences are given by

$$\Delta m = (\psi(\varphi), A\psi(\varphi)) \equiv (\phi, A\phi), \qquad \phi \in \bigoplus_{0}^{\infty} \mathscr{H}^{\otimes n} \tag{3.35}$$

and A is densely defined in the Fock space

$$\mathscr{F} = \bigoplus_{n=0}^{\infty} \mathscr{H}^{\otimes n}$$

This 'mass-breaking' operator ensures the existence of a mass formula, as has been worked out in von Westenholz (1971b). The structure of A may be specified as follows:

1. The eigenstates of A are given by the vectorvalued distributions (3.19) that is
$$A\psi^{ij}(\varphi) = \Delta m \psi^{ij}(\varphi) \tag{3.36}$$
2. By (3.35) A must be constant with respect to a cohomology class for some fixed ϕ because according to (3.28) and (3.29) one has the bijective assignment
$$\Delta m_{ij} \neq 0 \leftrightarrow \{f\}_{ij} \in H^2(P_+{}^{\uparrow}, G_1) \tag{3.37}$$
Therefore we write: $A = C(f)$ (C stands for 'constant' and f denotes a representative $\in \{f\} \in H^2(P_+{}^{\uparrow}, G_1)$. Thus we have
$$C: H^2(P_+{}^{\uparrow}, G_1) \xrightarrow{\text{into}} \mathscr{L}\left(\bigoplus_{n=0}^{\infty} \mathscr{H}^{\otimes n}, \bigoplus_{n=0}^{\infty} \mathscr{H}^{\otimes n}\right) \tag{3.38}$$

and obviously

$$f_1 \nsim f_2 \bmod B^2(P_+{}^{\uparrow}, G_1) \Leftrightarrow C(f_1) \neq C(f_2) \tag{3.39}$$

In particular one has the following

Lemma 4

If the map (3.38) constitutes a linear operator in the Fock space

$$\mathscr{F} = \bigoplus_{n=0}^{\infty} \mathscr{H}^{\otimes n}$$

then

$$C(0) = 0 \Leftrightarrow \{f\} = \{0\} \in H^2(P_+{}^{\uparrow}, G_1) \tag{3.40}$$

must hold.

Proof: Indeed, for $f \sim f' \bmod B^2$: $C(f) = C(f') = C(f+b)$, since C is constant on each cohomology class. Moreover, $C(f+b) = C(f) + C(b)$ by linearity of C and therefore

$$C(b) = 0 \tag{3.41}$$

On account of formula (2.6) one has to characterise mass degeneracy by equation

$$[P^\mu, X_k] = 0 \Leftrightarrow \Delta \cdot m_{ik} = 0 \qquad \forall i, k \Leftrightarrow C(f_0) = 0 \tag{2.6'}$$

Therefore $f_0 \in B^2(P_+{}^\uparrow, G_1)$, that is $f_0 \sim 0 \bmod B^2(P_+{}^\uparrow, G_1)$.

Thus one obtains the following corollary to Lemma 2:

Corollary: Vanishing mass differences of some multiplet must be associated with the cohomology class $\{0 + B^2(P_+{}^\uparrow, G_1)\}$, i.e.

$$\Delta m_{ik} = 0 \leftrightarrow \{0\} \in H^2(P_+{}^\uparrow, G_1) \tag{3.42}$$

These results suggest a description of a modified classification scheme for elementary particles. In fact, one infers by inspection of the correspondences (3.29) and (3.37) that such a symmetry scheme requires an appropriate choice of some group G_2 such that

$$H^2(P_+{}^\uparrow, G_1) \cong G_2 \tag{3.43}$$

where the order of G_2 must equal $(n+1)$ (n denotes the number of non-vanishing mass differences of some given particle family).

To summarise: Non-trivial mixings between $P_+{}^\uparrow$ and some internal symmetry must be related to some modified classification scheme, which is associated with

(a) A non-compact and non-semisimple internal symmetry G_1 (in general non-abelian)
(b) The symmetry P, which acts on space-time degrees of freedom
(c) Some 'classification group' G_2 (this will in general be a discrete group, e.g. $\mathbf{Z}_n$) and
(d) A mass-breaking operator

$$C(f) \in \mathscr{L}(\mathscr{F}, \mathscr{F})$$

(3.44)

Remark 4: By virtue of (3.44a) one can construct mass formulae of the type

$$m = m_0 + \Delta m \tag{3.45}$$

where

$$\Delta m = \phi(X_1, X_2, \ldots, X_n); \qquad X_k \in \mathscr{H} \text{ (the internal Lie algebra)}$$

This has been shown in detail in von Westenholz (1971b).

Discussion of the modified classification scheme

Consider a J^P-family of strongly interacting particles (J: spin, P: parity) whose masses are $m_1 < m_2 \ldots < m_n$; $m_i \in \bar{V}^+ = \{p \in \mathbf{R}^4, p^2 \geqslant 0, p^0 \geqslant 0\}$

(i.e. there exist $(n-1)$ independent mass differences $\Delta m_{ij} \neq 0$). According to our scheme (3.44), we seek some internal symmetry G_1 which displays the following properties:

$$\left.\begin{array}{ll}
\text{(a)} & G_1 \text{ is a separable locally compact, non semi-simple and non-abelian symmetry} \\
\text{(b)} & \text{The centre of } G_1, C(G_1) \text{ is such that } H^2(P_+{}^{\uparrow}, C(G_1)) = G_2 \\
\text{(c)} & G_1 \text{ possesses a maximal closed compact subgroup } G_3, \text{ such that } G_3 \text{ has an irreducible representation the dimension of which equals the order of } G_2 \\
\text{(d)} & \text{The representations of } G_1 \text{ are built up as induced representations (Mackey, 1955) from the representations of } G_3. \text{ In particular, these are (up to unitary equivalence) identical with (3.21)}
\end{array}\right\} \quad (3.45)$$

The order of G_2 shall be determined by the number of mass differences of the particle masses of the particles within some super-multiplet. It equals n, if the given multiplet is completely filled and $n' > n$ otherwise (i.e. there would be $n' - n$ vacant members within the multiplet). We therefore distinguish the following two cases:

(1) The super-multiplet is completely filled and one can find, in principle, a group G_1 which exhibits the properties 63a–d). It must be stressed at once that the construction of such a group certainly requires a considerable mathematical apparatus. We shall attempt in a forthcoming paper to carry through such a programme.

(2) The super-multiplet is not filled, one therefore cannot find any group with the properties (3.45a–d) as long as n is the order of G_2. Then one would have to assume that the order of G_2 equals $(n+1)$. If, under this condition, the group G_1 can be determined, one has to conclude that there was a member missing in the initial super-multiplet. Otherwise one has to assume that the order of G_2 is greater than $(n+1)$ and one has to carry on with this procedure until, for a certain k, the order of G_2 is $n' = n + k$ and permits the complete determination of the symmetry G via (3.45a–d). This means that k members were missing in the starting multiplet. $U(G_3)$ then classifies these n' particles.

At the classificatory level of the conventional models, such as the 'eightfold way', one has no clear indication of what the underlying symmetry group actually is (there exists for instance three non-isomorphic rank-two groups). Furthermore, once a group has been chosen, it is not clear which are the correct assignments of the particles to its irreducible representations. In the case of our model (3.45) the right choice of the relevant symmetry should emerge from a comparison of the theoretical predictions with experiments, since the $(n-1)$ mass differences Δm_{ij} should determine G_1.

The above-mentioned discussion applies to the case of some set of strongly interacting particles. By switching on some weaker interaction, one obtains the following interesting result (von Westenholz, 1971c).

Theorem 2

Let E be a Lie group-symmetry which contains the inhomogeneous Lorentz group $P_+{}^\uparrow$ and some internal symmetry group G ($\dim G = n$) as analytic subgroups. G is supposed to be an exact symmetry for some strong interaction model, i.e. $E = G \times P_+{}^\uparrow$. Let the spectrum of the 'mass-breaking' operator $C(f)$ and of the energy momentum vector P be in $\{0\} \cup \bar{V}^+$. Then the breakdown of the symmetry G may be characterised by the following equivalent statements:

(a) The trivial direct product-coupling $E = G \times P_+{}^\uparrow$ is carried over in nontrivial mixings $E_\alpha = G \perp_\alpha P_+{}^\uparrow$ ($\perp_\alpha$ symbolises these mixings).
(b) There remains invariance of the theory only under transformations corresponding to a group G_1 of smaller dimension ($\dim G_1 = n_1 > n$).
(c) The mass degeneracy is (partly) removed.

The proof of this Theorem can be found in von Westenholz (1971c). This Theorem constitutes actually a rigorous characterisation of broken symmetries as described by relationship (2.2) in our Section 2.

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